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# A Note on the Hyers–Ulam Stability Constants of Closed Linear Operators

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**Abstract.** This paper concerns the properties of the Hyers–Ulam stability constant of closed linear operators. Using the Moore–Penrose inverse, we prove that the mapping $\overline{T} \to K_{\overline{T}}$ is lower semi-continuous and give some sufficient and necessary conditions for $\overline{T} \to K_{\overline{T}}$ to be continuous or locally bounded.

## 1. Introduction and Preliminaries

In 1940, S. M. Ulam [16], in a talk given at Wisconsin University, posed the well-known stability problem, which was partially solved by D. H. Hyers [9] in the framework of Banach spaces. Due to the question of Ulam and the answer of Hyers the stability of equations is called after their names. Later, a large number of papers and books have been published in connection with various generalizations of Hyers–Ulam theorem[5, 7, 10, 13, 15, 17]. For instance, S.-E. Takahasi, H. Takagi, T. Miura and S. Miyajima [15] investigated the Hyers–Ulam stability constant $K_{T_h}$ of linear differential operator $(T_h u)(t) = u'(t) + h(t)u(t)$ and pointed out that it would be interesting to investigate the properties of the mapping $h \to K_{T_h}$.

Motivated by this and the fact that the differential operators are always closed linear operators, we investigate the properties of the mapping $\overline{T} \to K_{\overline{T}}$ for closed linear operators in this paper. The aim of this work is to prove that $\overline{T} \to K_{\overline{T}}$ is lower semi-continuous and provide some sufficient and necessary conditions for $\overline{T} \to K_{\overline{T}}$ to be continuous or locally bounded. To achieve our results, we need some terminology as follows.

Let $X$,$Y$ be Hilbert spaces. Let $C(X, Y)$ and $B(X, Y)$ denote the homogeneous set of all closed linear operators with a dense domain and the Banach space of all bounded linear operators from $X$ into $Y$, respectively. The identity operator is denoted by $I$.

**Definition 1.1.** *[6] Let $T$ be a (not necessarily linear) mapping from the Domain $D(T) \subset X$ into $Y$. We say that $T$ has the Hyers–Ulam stability if there exists a constant $K > 0$ with the property: For any $y$ in the range $R(T)$ of $T$, $\varepsilon > 0$ and $x \in D(T)$ with $\|Tx - y\| \leq \varepsilon$, there exists $x_0 \in D(T)$ such that $Tx_0 = y$ and $\|x - x_0\| \leq K\varepsilon$. We call such $K > 0$ a Hyers–Ulam stability constant for $T$ and denote by $K_T$ the infimum of all Hyers–Ulam stability constants for $T$. If $K_T$ is a Hyers–Ulam stability constant for $T$, then $K_T$ is called the Hyers–Ulam stability constant for $T$.*

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Roughly speaking, if $T$ has the Hyers–Ulam stability, then to each $\varepsilon$-approximate solution $x$ of the equation $Tx = y$, there corresponds an exact solution $x_0$ of the equation in a $K\varepsilon$-neighborhood of $x$ [6].

**Definition 1.2.** *[1] An operator $T \in C(X, Y)$ possesses a (bounded) generalized inverse if there exists an operator $S \in B(Y, X)$ such that $R(S) \subseteq D(T)$ and (1) $TSTx = Tx$ for all $x \in D(T)$; (2) $STSy = Sy$ for all $y \in Y$; (3) $ST$ is continuous. We denote a generalized inverse of $T$ by $T^+$.*

An operator $T \in C(X, Y)$ has a generalized inverse $T^+ \in B(Y, X)$ if and only if the null space $N(T)$ has a topological complement $N(T)^c$ in $X$ and $R(T)$ has a topological complement $R(T)^c$ in $Y$, i.e.,

$$X = N(T) \oplus N(T)^c \quad \text{and} \quad Y = R(T) \oplus R(T)^c. \tag{1.1}$$

From the closed graph theorem, it follows that the operator $TT^+$ is a projector from $Y$ onto $R(T)$ such that $N(TT^+) = N(T^+)$ and $R(TT^+) = R(T)$. Meanwhile, $T^+T$ can be extended uniquely to a projector from $X$ onto $\overline{R(T^+)}$ with the null space $N(T)$ and the range $\overline{R(T^+)}$ [14].

**Definition 1.3.** *[14] If the topological decompositions in (1.1) are orthogonal, i.e.,*

$$X = N(T) \dotplus N(T)^\perp \quad \textit{and} \quad Y = R(T) \dotplus R(T)^\perp,$$

*where $\dotplus$ denotes the orthogonal direct sum, then the corresponding generalized inverse is called the Moore–Penrose inverse of $T$, which is usually denoted by $T^\dagger$.*

A relationship between the Hyers–Ulam stability and the Moore–Penrose inverse of closed operators is established in [7].

**Theorem 1.4.** *[7] Let $T \in C(X, Y)$, then the following statements are equivalent:*
*(1) $T$ has the Hyers–Ulam stability;*
*(2) $T$ has the bounded Moore–Penrose inverse $T^\dagger$;*
*(3) $T$ has a bounded generalized inverse $T^+$;*
*(4) $T$ has a closed range.*

*Moreover, if one of the conditions above is true, then $N(T^\dagger) = R(T)^\perp$, $R(T^\dagger) = D(T) \cap N(T)^\perp$, $T^\dagger = [I - P^\perp_{N(T)}]T^+P^\perp_{R(T)}$ and the Hyers–Ulam stability constant*

$$K_T = \|T^\dagger\|.$$

## 2. Main Results

**Lemma 2.1.** *[19] Let $S$ be a densely defined and bounded linear operator from $D(S) \subset X$ into $Y$, then there exists a unique norm-preserving extension $W : X \to Y$ of $S$ such that $D(W) = X$,*

$$W^* = S^* \quad \text{and} \quad W = S^{**}.$$

**Theorem 2.2.** *Let $T \in C(X, Y)$ have the Hyers–Ulam stability, i.e, $T$ has a Moore–Penrose inverse $T^\dagger \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta T\|\|T^\dagger\| < \frac{1}{3}(3 + 2\sqrt{3})$, then $\overline{T} = T + \delta T \in C(X, Y)$ and the function*

$$f : \overline{T} \to K_{\overline{T}}$$

*is lower semi-continuous at $T$.*

*Proof.* Noting $T \in C(X, Y)$ and $\delta T \in B(X, Y)$, it is easy to verify $\overline{T} = T + \delta T \in C(X, Y)$. Because to consider the lower semi-continuity, we can suppose that $K_{\overline{T}}$ is finite. Then by Theorem 1.1 and Lemma 2.1, $\overline{T}$ has the Moore-Penrose inverse $\overline{T}^\dagger$ and $\overline{T}\,\overline{T}^\dagger$ is the orthogonal projector from $Y$ onto $R(\overline{T})$, $(\overline{T}^\dagger\overline{T})^*$ is the norm-preserving extension of $\overline{T}^\dagger\overline{T}$ and is exactly the orthogonal projector from $X$ onto $\overline{R(\overline{T}^\dagger)}$. Noting

$R(T^\dagger) \subset D(T) = D(\overline{T})$ and $R((I - \overline{T}^\dagger\overline{T})^*) = \overline{R(I - \overline{T}^\dagger\overline{T})} = \overline{N(\overline{T})} = N(\overline{T}) \subset D(\overline{T}) = D(T)$, we can obtain $(\overline{T}^\dagger\overline{T} - I)^*T^\dagger = (\overline{T}^\dagger\overline{T} - I)T^\dagger$ and $(T^\dagger T)^*(\overline{T}^\dagger\overline{T} - I)^* = (T^\dagger T)(\overline{T}^\dagger\overline{T} - I)^*$, and so

$$\begin{aligned}
\overline{T}^\dagger - T^\dagger &= \overline{T}^\dagger(I - TT^\dagger) + \overline{T}^\dagger(T - \overline{T})T^\dagger + (\overline{T}^\dagger\overline{T} - I)T^\dagger \\
&= \overline{T}^\dagger\,\overline{T}\,\overline{T}^\dagger(I - TT^\dagger) + \overline{T}^\dagger(T - \overline{T})T^\dagger + (\overline{T}^\dagger\,\overline{T} - I)T^\dagger TT^\dagger \\
&= \overline{T}^\dagger(\overline{T}\,\overline{T}^\dagger)^*(I - TT^\dagger)^* + \overline{T}^\dagger(T - \overline{T})T^\dagger + (\overline{T}^\dagger\,\overline{T} - I)^*T^\dagger TT^\dagger \\
&= \overline{T}^\dagger(\overline{T}\,\overline{T}^\dagger)^*(I - TT^\dagger)^* + \overline{T}^\dagger(T - \overline{T})T^\dagger + (\overline{T}^\dagger\,\overline{T} - I)^*(T^\dagger T)^*T^\dagger \\
&= \overline{T}^\dagger[(I - TT^\dagger)(\overline{T}\,\overline{T}^\dagger)]^* + \overline{T}^\dagger(T - \overline{T})T^\dagger + [(T^\dagger T)^*(\overline{T}^\dagger\overline{T} - I)^*]^*T^\dagger \\
&= \overline{T}^\dagger[(I - TT^\dagger)(\overline{T}\,\overline{T}^\dagger)]^* + \overline{T}^\dagger(T - \overline{T})T^\dagger + [(T^\dagger T)(\overline{T}^\dagger\overline{T} - I)^*]^*T^\dagger \\
&= \overline{T}^\dagger[(I - TT^\dagger)(T + \delta T)\,\overline{T}^\dagger]^* - \overline{T}^\dagger\delta TT^\dagger + [(T^\dagger\overline{T} - T^\dagger\delta T)(\overline{T}^\dagger\overline{T} - I)^*]^*T^\dagger \\
&= \overline{T}^\dagger[(I - TT^\dagger)\delta T\overline{T}^\dagger]^* - \overline{T}^\dagger\delta TT^\dagger + [(T^\dagger\delta T)(I - \overline{T}^\dagger\overline{T})^*]^*T^\dagger \\
&= \overline{T}^\dagger(\overline{T}^\dagger)^*(\delta T)^*(I - TT^\dagger) - \overline{T}^\dagger\delta TT^\dagger + (I - \overline{T}^\dagger\overline{T})^*(\delta T)^*(T^\dagger)^*T^\dagger,
\end{aligned}$$

i.e.,

$$\overline{T}^\dagger = T^\dagger + \overline{T}^\dagger(\overline{T}^\dagger)^*(\delta T)^*(I - TT^\dagger) - \overline{T}^\dagger\delta TT^\dagger + (I - \overline{T}^\dagger\overline{T})^*(\delta T)^*(T^\dagger)^*T^\dagger. \tag{2.1}$$

Therefore

$$\|\overline{T}^\dagger\| \geq \|T^\dagger\| - \|\delta T\| \cdot \|\overline{T}^\dagger\|^2 - \|\overline{T}^\dagger\| \cdot \|\delta T\| \cdot \|T^\dagger\| - \|\delta T\| \cdot \|T^\dagger\|^2.$$

Set $p = \|\delta T\|, q = \|T^\dagger\|$, then

$$pK_{\overline{T}}^2 + (1 + pq)K_{\overline{T}} + pq^2 - q \geq 0. \tag{2.2}$$

Since $0 < pq = \|\delta T\|\|T^\dagger\| < \frac{1}{3}(3 + 2\sqrt{3})$, we get that the discriminant is

$$\begin{aligned}
\triangle &= (1 + pq)^2 - 4p(pq^2 - q) \\
&= -3p^2q^2 + 6pq + 1 \\
&= -3[pq - \frac{1}{3}(3 + 2\sqrt{3})][pq - \frac{1}{3}(3 - 2\sqrt{3})] > 0.
\end{aligned}$$

It follows from (2.2) that

$$K_{\overline{T}} \geq \frac{-(1 + pq) + \sqrt{\triangle}}{2p} = \frac{2q - 2pq^2}{1 + pq + \sqrt{-3p^2q^2 + 6pq + 1}}. \tag{2.3}$$

Let $p = \|\delta T\| \to 0$, then the right formula in (2.3) approaches to $q = K_T$. Thus

$$\liminf_{\|\delta T\|\to 0} K_{\overline{T}} \geq K_T.$$

The proof is completed. □

**Theorem 2.3.** *Let $T \in C(X, Y)$ have the Hyers–Ulam stability, i.e, $T$ has a Moore-Penrose inverse $T^\dagger \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta T\| \cdot \|T^\dagger\| < \frac{1}{3}(3 + 2\sqrt{3})$, then the following statements are equivalent:*
(1) *$\overline{T}$ has the Hyers–Ulam stability and the real valued function $f : \overline{T} \to K_{\overline{T}}$ is continuous at $T$, i.e.,*

$$\lim_{\|\delta T\|\to 0} K_{\overline{T}} = K_T;$$

(2) *$\overline{T}$ has the Hyers–Ulam stability and the real valued function $f : \overline{T} \to K_{\overline{T}}$ is locally bounded at $T$, i.e., there exist $M > 0$ and $\delta_1 > 0$ such that for all $\|\delta T\| < \delta_1$,*

$$K_{\overline{T}} \leq M.$$

*Proof.* Obviously, we need to show (2) $\Rightarrow$ (1). In fact, it follows from (2.1) that

$$\|\overline{T}^{\dagger}\| \leq \|T^{\dagger}\| + \|\overline{T}^{\dagger}\|^2\|\delta T\| + \|\overline{T}^{\dagger}\|\|T^{\dagger}\|\|\delta T\| + \|T^{\dagger}\|^2\|\delta T\|.$$

Then $\limsup_{\|\delta T\|\to 0} K_{\overline{T}} \leq K_T$. By Theorem 2.1, $\lim_{\|\delta T\|\to 0} K_{\overline{T}} = K_T$. The proof is completed. □

**Lemma 2.4.** *Let $M$ be a closed linear subspace of Hilbert space $X$. Let $P_M : X \to M$ be a (not necessarily selfadjoint) projector from $X$ onto $M$, then the orthogonal projector $P_M^{\perp}$ from $X$ onto $M$ can be expressed by*

$$P_M^{\perp} = -P_M(I - P_M - P_M^*)^{-1} = -(I - P_M - P_M^*)^{-1}P_M^*.$$

*Proof.* The projector $P_M$ is idempotent and so is $P_M^*$, then

$$(I - P_M - P_M^*)P_M = -P_M^*P_M = P_M^*(I - P_M - P_M^*). \tag{2.4}$$

Set $N = R(I - P_M) = N(P_M)$, then $N$ is a closed linear subspace of $X$ and

$$P_MP_M^{\perp} = P_M^{\perp},\ P_MP_N^{\perp} = P_N^{\perp}P_M = 0,\ (I - P_M)P_N^{\perp} = P_N^{\perp},\ P_N^{\perp}(I - P_M) = I - P_M.$$

Hence

$$\begin{aligned}(P_N^{\perp} - P_M^{\perp})(I - P_M - P_M^*) &= P_N^{\perp}(I - P_M) - P_M^{\perp}(I - P_M) - P_N^{\perp}P_M^* + P_M^{\perp}P_M^* \\ &= (I - P_M) - (P_M^{\perp} - P_M) - (P_MP_N^{\perp})^* + (P_MP_M^{\perp})^* \\ &= I - P_M^{\perp} + P_M^{\perp} = I.\end{aligned}$$

Similarly, $(I - P_M - P_M^*)(P_N^{\perp} - P_M^{\perp}) = I$. Thus $P_N^{\perp} - P_M^{\perp} = (I - P_M - P_M^*)^{-1} \in B(X)$ and

$$P_M^{\perp} = -P_M(P_N^{\perp} - P_M^{\perp}) = -P_M(I - P_M - P_M^*)^{-1}.$$

Utilizing (2.4), we have

$$P_M^{\perp} = -P_M(I - P_M - P_M^*)^{-1} = -(I - P_M - P_M^*)^{-1}P_M^*.$$

The proof is completed. □

**Lemma 2.5.** *Let $T \in C(X, Y)$ with a bounded generalized inverse $T^+ \in B(Y, X)$, then $T$ has the bounded Moore–Penrose inverse $T^{\dagger}$ and*

$$T^{\dagger} = [I - (T^+T)^* - (T^+T)^{**}]^{-1}T^+[I - TT^+ - (TT^+)^*]^{-1}.$$

*Proof.* Since $TT^+$ is a projector from $Y$ onto $R(T)$, by Lemma 2.2,

$$P_{R(T)}^{\perp} = -TT^+[I - TT^+ - (TT^+)^*]^{-1}.$$

By Lemma 2.1, $P_{N(T)} = (I - T^+T)^{**}$ is a projector from $X$ onto $N(T)$, $P_{N(T)}^* = (I - T^+T)^*$ and so

$$\begin{aligned}I - P_{N(T)}^{\perp} &= I + [I - P_{N(T)} - P_{N(T)}^*]^{-1}P_{N(T)}^* \\ &= [I - P_{N(T)} - P_{N(T)}^*]^{-1}[I - P_{N(T)}] \\ &= [I - (I - T^+T)^{**} - (I - T^+T)^*]^{-1}[I - P_{N(T)}] \\ &= -[I - (T^+T)^* - (T^+T)^{**}]^{-1}[I - P_{N(T)}].\end{aligned}$$

It follows from $P_{N(T)}|_{D(T)} = I - T^+T$ that $[I - P_{N(T)}]T^+ = T^+$ and

$$\begin{aligned}T^{\dagger} &= [I - P_{N(T)}^{\perp}]T^+P_{R(T)}^{\perp} \\ &= [I - (T^+T)^* - (T^+T)^{**}]^{-1}[I - P_{N(T)}]T^+TT^+[I - TT^+ - (TT^+)^*]^{-1} \\ &= [I - (T^+T)^* - (T^+T)^{**}]^{-1}T^+[I - TT^+ - (TT^+)^*]^{-1}.\end{aligned}$$

The proof is completed. □

Next, we shall give some sufficient and necessary conditions for the mapping of the Hyers-Ulam stability constants to be continuous or locally bounded.

**Theorem 2.6.** *Let $T \in C(X, Y)$ have the Hyers–Ulam stability, i.e, $T$ has a Moore-Penrose inverse $T^{\dagger} \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta T\| \cdot \|T^{\dagger}\| < 1$, then the following statements are equivalent:*
(1) *$B = T^{\dagger}(I + \delta T T^{\dagger})^{-1} = (I + T^{\dagger}\delta T)^{-1}T^{\dagger} : Y \to X$ is a generalized inverse of $\overline{T}$;*
(2) *$R(\overline{T}) \cap N(T^{\dagger}) = \{0\}$;*
(3) *$(I + \delta T T^{\dagger})^{-1}\overline{T}$ maps $N(T)$ into $R(T)$;*
(4) *$\overline{T}$ has the Moore-Penrose inverse $\overline{T}^{\dagger} \in B(Y, X)$ with $\lim_{\|\delta T\| \to 0} \overline{T}^{\dagger} = T^{\dagger}$;*
(5) *$\overline{T}$ has the Hyers–Ulam stability and the real valued function $f : \overline{T} \to K_{\overline{T}}$ is continuous at $T$, i.e.,*

$$\lim_{\|\delta T\| \to 0} K_{\overline{T}} = K_T;$$

(6) *$\overline{T}$ has the Hyers–Ulam stability and the real valued function $f : \overline{T} \to K_{\overline{T}}$ is locally bounded at $T$, i.e., there exist $M > 0$ and $\delta_1 > 0$ such that for all $\|\delta T\| < \delta_1$,*

$$K_{\overline{T}} \le M.$$

*In this case, the Hyers–Ulam stability constant $K_{\overline{T}} = \|\overline{T}^{\dagger}\|$ and*

$$\begin{aligned} \overline{T}^{\dagger} &= \{I - [(I + T^{\dagger}\delta T)^{-1}T^{\dagger}\overline{T}]^* - [(I + T^{\dagger}\delta T)^{-1}T^{\dagger}\overline{T}]^{**}\}^{-1} \\ &\quad T^{\dagger}(I + \delta T T^{\dagger})^{-1}\{I - [\overline{T}T^{\dagger}(I + \delta T T^{\dagger})] - [(\overline{T}T^{\dagger}(I + \delta T T^{\dagger})^{-1}]^*\}^{-1}. \end{aligned}$$

*Proof.* It follows from Theorem 2.1 in [8] that (1) ⇔ (2) ⇔ (3) and obviously, (4) ⇒ (5) ⇔ (6). To the end, we shall show (1) ⇒ (4) and (5) ⇒ (4) ⇒ (3).

(1) ⇒ (4). If $B$ is a generalized inverse of $\overline{T}$, then by Lemma 2.5, $\overline{T}$ has the Moore-Penrose inverse

$$\begin{aligned} \overline{T}^{\dagger} &= [I - (B\overline{T})^* - (B\overline{T})^{**}]^{-1}B[I - (\overline{T}B) - (\overline{T}B)^*]^{-1} \\ &= \{I - [(I + T^{\dagger}\delta T)^{-1}T^{\dagger}\overline{T}]^* - [(I + T^{\dagger}\delta T)^{-1}T^{\dagger}\overline{T}]^{**}\}^{-1} \\ &\quad T^{\dagger}(I + \delta T T^{\dagger})^{-1}\{I - [\overline{T}T^{\dagger}(I + \delta T T^{\dagger})^{-1}] - [\overline{T}T^{\dagger}(I + \delta T T^{\dagger})^{-1}]^*\}^{-1}. \end{aligned}$$

Noticing

$$\begin{aligned} [(I + T^{\dagger}\delta T)^{-1}T^{\dagger}\overline{T}]^* &= [(I + T^{\dagger}\delta T)^{-1}T^{\dagger}T + (I + T^{\dagger}\delta T)^{-1}T^{\dagger}\delta T]^* \\ &= (T^{\dagger}T)^*[(I + T^{\dagger}\delta T)^{-1}]^* + (\delta T)^*(T^{\dagger})^*[(I + T^{\dagger}\delta T)^{-1}]^* \to (T^{\dagger}T)^* \end{aligned}$$

and $\overline{T}T^{\dagger}(I + \delta T T^{\dagger})^{-1} = TT^{\dagger}(I + \delta T T^{\dagger})^{-1} + \delta T T^{\dagger}(I + \delta T T^{\dagger})^{-1} \to TT^{\dagger}$, we can see

$$\overline{T}^{\dagger} \to [I - (T^{\dagger}T)^* - (T^{\dagger}T)^{**}]^{-1}T^{\dagger}[I - TT^{\dagger} - (TT^{\dagger})^*]^{-1} = T^{\dagger}.$$

(5) ⇒ (4). It follows from (2.1) that

$$\|\overline{T}^{\dagger} - T^{\dagger}\| \le (\|\overline{T}^{\dagger}\|^2 + \|\overline{T}^{\dagger}\| \cdot \|T^{\dagger}\| + \|T^{\dagger}\|^2)\|\delta T\|.$$

Combining it with (5), we can conclude $\lim_{\|\delta T\| \to 0} \overline{T}^{\dagger} = T^{\dagger}$.

(4) ⇒ (3). If $\overline{T}$ has the Moore–Penrose inverse $\overline{T}^{\dagger}$ with $\lim_{\|\delta T\| \to 0} \overline{T}^{\dagger} = T^{\dagger}$, then, by $P^{\perp}_{N(\overline{T})} = (I - \overline{T}^{\dagger}\overline{T})^*$, $P^{\perp}_{N(T)} = (I - T^{\dagger}T)^*$ and

$$\begin{aligned} P^{\perp}_{N(\overline{T})} - P^{\perp}_{N(T)} &= (I - \overline{T}^{\dagger}\overline{T})^* - (I - T^{\dagger}T)^* = (T^{\dagger}T - \overline{T}^{\dagger}\overline{T})^* \\ &= -[\overline{T}^{\dagger}(\overline{T} - T) + (\overline{T}^{\dagger} - T^{\dagger})T]^* \\ &= -(\overline{T}^{\dagger}\delta T)^* + [\overline{T}^{\dagger}\delta T T^{\dagger}T - (I - \overline{T}^{\dagger}\overline{T})^*(\delta T)^*(T^{\dagger})^*T^{\dagger}T]^* \\ &= -(I - T^{\dagger}T)^*(\overline{T}^{\dagger})^*(\delta T)^* - (T^{\dagger}T)^*T^{\dagger}\delta T(I - \overline{T}^{\dagger}\overline{T})^*, \end{aligned}$$

we can obtain $\lim_{\|\delta T\|\to 0} P^{\perp}_{N(\overline{T})} = P^{\perp}_{N(T)}$. Without loss of generality, we may assume $\|P^{\perp}_{N(\overline{T})} - P^{\perp}_{N(T)}\| < 1$. From [11], $R(P^{\perp}_{N(T)}) = P^{\perp}_{N(T)}R(P^{\perp}_{N(\overline{T})})$, i.e.,

$$N(T) = P^{\perp}_{N(T)}N(\overline{T}) = (I - T^{\dagger}T)^{*}N(\overline{T}) = (I - T^{\dagger}T)N(\overline{T}).$$

Hence, for any $x \in N(T)$, there is an element $x_1 \in N(\overline{T})$ such that $x = (I - T^{\dagger}T)x_1$. Therefore,

$$\begin{aligned}
(I + \delta TT^{\dagger})^{-1}\overline{T}x &= (I + \delta TT^{\dagger})^{-1}\overline{T}(I - T^{\dagger}T)x_1 \\
&= (I + \delta TT^{\dagger})^{-1}\overline{T}T^{\dagger}T(-x_1) \\
&= (I + \delta TT^{\dagger})^{-1}(T + \delta T)T^{\dagger}T(-x_1) \\
&= (I + \delta TT^{\dagger})^{-1}(I + \delta TT^{\dagger})T(-x_1) \\
&= T(-x_1) \in R(T),
\end{aligned}$$

which implies the statement (3) holds. The proof is completed. □

**Remark 2.7.** *Theorem 2.3 is a direct generalization of Theorem 3.1 in [7] to the case of closed linear operators. It's also worth pointing out that the expression of Moore–Penrose inverse in Theorem 2.3 is more concise than the ones in [2, 7, 8, 12, 18]. Note that the expression in statement (1) maybe the simplest possible one for the generalized inverse [4], the statement (2) is called to be a stable perturbation of T [3] and the statement (3) is first discovered by M. Z. Nashed to be a condition for generalized inverse to be stable [14].*

**Corollary 2.8.** *Let $T \in C(X, Y)$ have the Hyers–Ulam stability, i.e, $T$ has a Moore-Penrose inverse $T^{\dagger} \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta T\| \cdot \|T^{\dagger}\| < 1$,*

$$N(T) \subseteq N(\delta T) \quad \text{or} \quad R(\delta T) \subseteq R(T),$$

*then $\overline{T}$ has the Hyers–Ulam stability and $\lim_{\|\delta T\|\to 0} K_{\overline{T}} = K_T$.*

*Proof.* If $N(T) \subset N(\delta T)$, then $N(T) \subset N(\overline{T})$. By the statement (3) in Theorem 2.6, we can get what we desired. If $R(\delta T) \subset R(T)$, then $R(\overline{T}) \subset R(T)$. Noting $R(T) \cap N(T^{+}) = \{0\}$ and the statement (2) in Theorem 2.6, we can complete the proof. □

**Corollary 2.9.** *Let $X$, $Y$ be Hilbert spaces and let $T \in C(X, Y)$ be a semi-Fredholm operator. If $T$ has the bounded Moore–Penrose inverse $T^{\dagger} \in B(Y, X)$ and $\delta T \in B(X, Y)$ satisfies $\|\delta T\|\|T^{\dagger}\| < 1$, then $\overline{T} = T + \delta T$ has the Hyers–Ulam stability and the Hyers–Ulam stability constant $K_{\overline{T}}$ satisfies $\lim_{\|\delta T\|\to 0} K_{\overline{T}} = K_T$ if and only if*

$$\textit{either } \dim N(\overline{T}) = \dim N(T) < +\infty \textit{ or } \operatorname{codim} R(\overline{T}) = \operatorname{codim} R(T) < +\infty.$$

*Proof.* The proof of the sufficiency is similar to Theorem 2.12 in [7], we omit it. In the following, we shall show the necessity. If $\overline{T}$ has the Hyers–Ulam stability and $\lim_{\|\delta T\|\to 0} K_{\overline{T}} = K_T$, then by Theorem 2.6, $B = T^{\dagger}(I + \delta TT^{\dagger})^{-1} = (I + T^{\dagger}\delta T)^{-1}T^{\dagger} : Y \to X$ is a generalized inverse of $\overline{T}$. Thus

$$\overline{R(T^{\dagger})} \dotplus N(T) = X = \overline{R(B)} \oplus N(\overline{T}) = \overline{R(T^{\dagger})} \oplus N(\overline{T}) \tag{2.5}$$

and

$$R(T) \dotplus N(T^{\dagger}) = Y = R(\overline{T}) \oplus N(B) = R(\overline{T}) \oplus N(T^{\dagger}). \tag{2.6}$$

If $\dim N(T) < +\infty$, then by (2.5), $\operatorname{codim} \overline{R(T^{\dagger})} = \dim N(T) < +\infty$ and

$$\dim N(\overline{T}) = \operatorname{codim} \overline{R(T^{\dagger})} = \dim N(T) < +\infty.$$

If $\operatorname{codim} R(T) < +\infty$, it follows from (2.6) that $\dim N(T^{\dagger}) = \operatorname{codim} R(T) < +\infty$ and therefore

$$\operatorname{codim} R(\overline{T}) = \dim N(T^{\dagger}) = \operatorname{codim} R(T) < +\infty.$$

The proof is completed. □

**Remark 2.10.** *Corollary 2.2 is a generalization of Theorem 3.3 in [7] to the case of closed linear operators.*

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